BOOK *of* INSTRUCTIONS

•

For the Use of

THE P.I.C.

Direct & Inverse Log-Log

DIFFERENTIAL

SCALES

for

$(1 + x)^n$ Expansions

and

Finance Calculations

Price One Shilling and Threepence

SUPPLEMENT TO THE P.I.C.
BOOK *of* INSTRUCTIONS

◆

Introducing the Use of
THE P.I.C.
Direct & Inverse Log-Log
DIFFERENTIAL
SCALES
for
Finance Calculations
of the order of

COMPOUNDING, AMORTIZATION OF ASSETS,
SINKING FUND ACCUMULATIONS, LOAN
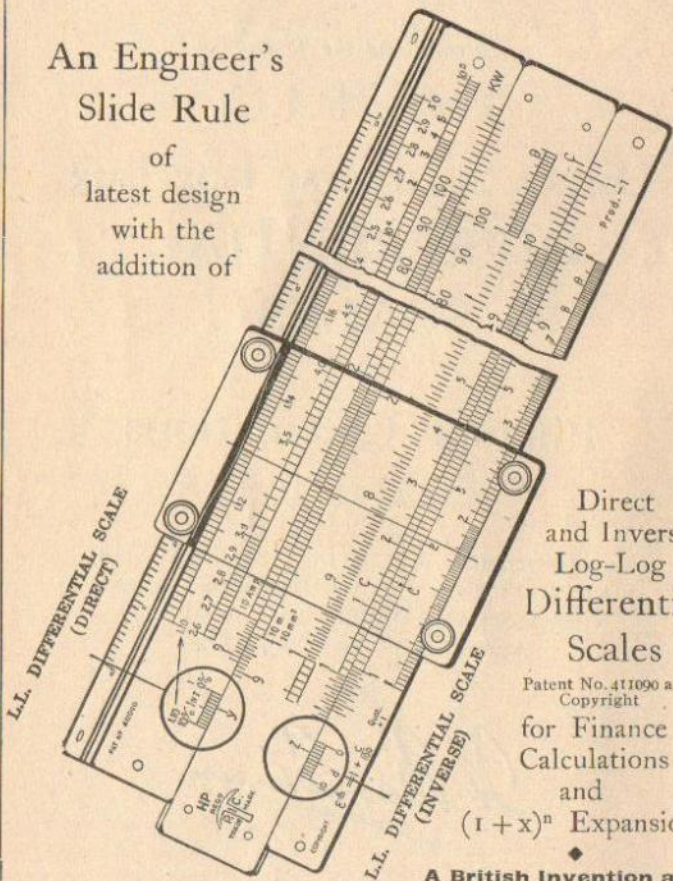CHARGES AND DEFERRED PAYMENTS,
ASSURANCE VALUES, ETC.

by

H. Boardman

THE "P.I.C."
LOG-LOG SLIDE RULE
(RANGE TO UNITY)

An Engineer's
Slide Rule
of
latest design
with the
addition of

Direct
and Inverse
Log-Log
Differential
Scales

Patent No. 411090 and
Copyright

for Finance
Calculations
and
$(1 + x)^n$ Expansions

◆

**A British Invention and
British Made**

L.L. DIFFERENTIAL SCALE
(DIRECT)

L.L. DIFFERENTIAL SCALE
(INVERSE)

FINANCIAL CALCULATIONS
SUPPLEMENT

Inspection of the following Mathematical
Formulæ concerning—

Compound Interest and Amortization; Sinking
Fund Accumulations; Loan Charges; Years Purchase
or Present Values of Future Incomes; Annuities,
Assurance Premiums, etc.; will immediately suggest
that evaluations of $\left(1 + \dfrac{R}{100}\right)^n$ form the major
portion of the computation work connected with the
fore-mentioned phases of Commercial Arithmetic.

In practice, interest rates rarely exceed 10% and so
the corresponding growth factors $\left(1 + \dfrac{R}{100}\right)$ which
have to be raised to the "n"th power usually fall
within the range $1\cdot0$ and $1\cdot10$.

The P.I.C. patented Log-Log Differential Scales,
subsequently described, specially facilitate this impor-
tant group of expansions and furnish values more
accurate than results obtainable by the use of four-
figure logarithmic tables.

Where R = Rate of Interest per cent.

n = number of years.

Then for—

Compound Growth.

The Value of £1 at R% Compound Interest after
n years

$$= \left(1 + \frac{R}{100}\right)^n \quad \cdots \cdots \cdots \quad (1)$$

Amortization or Compound Depreciation.

Where a series diminishes as a result of the successive applications of a constant factor less than unity, it is usually more convenient to consider a growth from the lower end and then to convert the growth percentage rate " R " to an amortization rate " D " by the form—

$$D = \frac{R}{1 + \frac{R}{100}} \cdot \cdot (2) \quad \text{or } vice\ versa \quad R = \frac{D}{1 - \frac{D}{100}} \cdot \cdot (2a)$$

This enables the form $\left(1 + \frac{R}{100}\right)^n$ to be retained, and eliminates the use of the form $\left(1 - \frac{D}{100}\right)^n$ which would involve negative logarithms or use of the reciprocal form in order to come within the range of the " 1 + " Log-log scale.

e.g. When 100 becomes 125, R = 25% (Growth)

but when 125 becomes 100, D = $\frac{25}{1\cdot25}$ = 20% (Depreciation)

Sinking Fund Accumulations.

The accumulated value of £1 per annum (end of year) payments for n consecutive years

$$= \frac{\left(1 + \frac{R}{100}\right)^n - 1}{\frac{R}{100}} \quad - \; - \; - \; - \; - \; - \; - \; - \; (3)$$

The equal annual end of year allocations to accumulate to £1 at the end of n years

= the reciprocal of (3)

ie.

$$= \frac{\frac{R}{100}}{\left(1 + \frac{R}{100}\right)^n - 1} \quad - \; - \; - \; - \; - \; - \; - \; - \; (3a)$$

Amount in a Sinking Fund.

The Amount due in a Sinking Fund after "y" years towards "S" to be accumulated in "n" years

$$= \frac{\left(1 + \dfrac{R}{100}\right)^{y} - 1}{\left(1 + \dfrac{R}{100}\right) - 1} \times S \quad \text{- - - - - - - (3b)}$$

Assurance.

Accumulated Value at the *end* of n years of £1 premiums paid at the *commencement* of each of n years

$$= \left\{ \frac{\left(1 + \dfrac{R}{100}\right)^{n+1} - 1}{\dfrac{R}{100}} \right\} - 1 \quad \text{- - - - - - (4)}$$

Loan Charges.

The equal annual (end of year) charges for each of n years to repay £1 of original Loan and Interest

or

The equal annual amounts extractable at the end of each of "n" years, from a commencement of period investment of £1

$$= \frac{\dfrac{R}{100}\left(1 + \dfrac{R}{100}\right)^{n}}{\left(1 + \dfrac{R}{100}\right)^{n} - 1} \quad \text{- - - - - - - (5)}$$

Note. In this connection, where L = Loan in £'s, P = total annual premium in £'s, and the determination of " R_1" or "n" is required, then the form $\left(1 + \dfrac{R}{100}\right)^{n} = \dfrac{P}{P - \dfrac{LR}{100}}$ will best serve.

Years' Purchase or Present Value.
(Single Rate)

Of £1 a year income due at the end of each of n consecutive years

$$= \text{The reciprocal of} \quad \cdots \cdots \cdots \quad (5)$$

$$= \frac{\left(1 + \frac{R}{100}\right)^n - 1}{\frac{R}{100}\left(1 + \frac{R}{100}\right)} \quad \cdots \cdots \cdots \quad (5a)$$

Years' Purchase or Present Value
(Dual Rate)

Applicable to the purchase of short lease property (n years to run) where the desired investment rate R and available sinking fund rate r differ

$$= \frac{100\left[\left(1 + \frac{r}{100}\right)^n - 1\right]}{R\left[\left(1 + \frac{r}{100}\right)^n - 1\right] + r} \quad \cdots \cdots \quad (6$$

Years' Purchase × Income = Capital $\cdots$ (7)

L.L. DIFFERENTIAL SCALE
DIRECT

LOG-LOG SCALES

←SLIDE

←C SCALE

←D SCALE

L.L. DIFFERENTIAL SCALE
(INVERSE)

CURSOR

NOTE.—

> *The subsequent paragraphs, examples, etc., of this booklet are supplementary to the descriptions and explanations of the uses of the normal Log-Log Scales given in pages 43 to 53 of the P.I.C. Book of Instructions. This preliminary knowledge is essential if the reader is to understand and appreciate what follows in regard to:*

Log-Log Differential Scales
Direct and Inverse

(Patent No. 411,090)

Normal Slide Rule practice in regard to Log-Log provision is to supply separate lengths of Log-Log Scale for the respective ranges 10^5 to $2 \cdot 7$ and $2 \cdot 7$ to $1 \cdot 1$.

These scales concern evaluations of "powers" and primarily serve where "Base" and "Expansion" fall within the range of 10^5 and $1 \cdot 1$, or where the reciprocals of "Base" and "Expansion" fall within the range 10^{-5} and $1 \cdot 1^{-1}$. Previously when dealing with numerical values outside these ranges, somewhat cumbersome factorising methods have had to be adopted. For the lower ranges such methods have pronounced limitations regarding significant figure accuracy as compared with results that can be obtained by means of Differential Scales.

Inasmuch as values of $\log_\varepsilon (1 + x)$ or expansions of the order ε^x (for x less than $0 \cdot 1$) are so often called for, also that "growth factors" $(1 + x)$ and their expansions $(1 + x)^n$, within the $(1 + x)$ range $1 \cdot 0$ and $1 \cdot 1$ so frequently occur in regard to Science and Finance computations, P.I.C. consider this field of sufficient importance to merit special consideration.

Slide Rule provision for such evaluations can be made by the addition of *either*

(i) Two additional full-length Log-Log scales for the respective ranges $1 \cdot 1$ to $1 \cdot 01$ and $1 \cdot 01$ to $1 \cdot 001$ (afterwards

referred to as the 1st and 2nd removes respectively).

or

(ii) Differential L.L. Scales.

In comparison, the Differential Scale system only requires $\frac{1}{40}$ of the additional scale length and does not appreciably disturb the general appearance of the rule; **it is this more scientific method that P.I.C. adopt.**

The Direct and Inverse L.L. Differential Scales together with the respective Indices Y and Z are clearly illustrated on page 4.

For values of $(1 + x)$ less than $1 \cdot 1$.

The Direct L.L. Differential Scale enables the cursor to be placed, in relation to the other log-log scales, to the log-log $(1 + x)$ position, and with the cursor so placed the evaluation of an expansion can easily be accomplished, also the significant figures of $\log_\varepsilon (1 + x)$ can be read direct from the D Scale.

To appreciate the **Inverse L.L. Scale** it must be remembered that where "d" has a value between 1 and 10 and the cursor is placed at "d" on the D Scale, the value of—

ε^d is the reading at the cursor on the lower log-log scale.

$\varepsilon^{\frac{d}{10}}$ is the reading at the cursor on the upper log-log scale.

The **Inverse L.L. Scale** enables the values $\varepsilon^{\frac{d}{100}}$ and $\varepsilon^{\frac{d}{1000}}$ to be obtained, or, *vice versa*, it enables the log-log readings in the $1 \cdot 10$ to $1 \cdot 01$ and $1 \cdot 01$ to $1 \cdot 001$ ranges to be obtained that correspond to any cursor position.

The following examples will serve to illustrate the uses of the Direct and Inverse L.L. Differential Scales :

(Note.—As regards $(1 + x)^n$, where "x" is less than $0 \cdot 1$ and "nx" not greater than unity, the following rough approximation will serve to fix the region of the value of the expansion, *i.e.*, the selection of the range in which the desired value will occur.

$(1 + x)^n$ **approximates to, but is greater than $(1 + nx)$.**

Example 1.—To determine the Amount (A) of £1 after 16 years at 4 per cent. per annum compound interest.

$$A = \left(1 + \frac{R}{100}\right)^n = \left(1 + \cdot04\right)^{16}$$

To evaluate $(1\cdot04)^{16}$

Evaluate "*nx*" (usually, as in this case, it can be done by mental arithmetic—otherwise use the C and D Scales)

$$nx = 0\cdot64$$

[By rough approximation, $(1\cdot04)^{16} \doteq 1\cdot64$. Hence the desired value will occur on the upper Log-log scale.]

For accurate determination, set the Index Y at 4 per cent. (1·04) on the Direct Log-log Differential Scale and the cursor to C64. Read the value of A at the cursor on the upper Log-log scale,

viz. : **1·873** *i.e.,*

$$A = £1 \ 17s. \ 5\frac{1}{2}d.$$

Alternative method—

1. Move the slide so that the index Y is at 4% on the Differential Scale.

2. Set the cursor at 4 on the C Scale.

3. Move the 1 (or 10) of the C Scale to the cursor.

4. Move the cursor to the number of years on the C Scale *i.e.,* 16.

5. Read the value of $\left(1 + \dfrac{4}{100}\right)^{16}$ on the appropriate Log-log Scale *i.e.,* 1·873. *Answer £1·873.*

It will be observed that after (3), values for periods 10 to 25 years can be projected direct from the C Scale on to the upper portion of the Log-log Scale, and by reversing the slide, *i e.,* 10 on the C Scale to the cursor position after (2), values for periods 25 to 100 years are then readable on the lower Log-log Scale.

See example 4 for normal rates of interest applicable to short periods (*i.e.,* where R × n is less than 10). Also example 4a, for abnormal rates (*i.e.,* interest rates in excess of 10%).

Example 1b.

What *Principal* **invested at 4% Compound Interest will amount to £49 in 16 years ?**

$$\text{Here Principal} = \frac{£49}{\text{Value obtained in Ex. 1.}} = \frac{49}{1\cdot 873}$$

Carry out this division on the C and D Scales as explained on pages 9 to 12 of the general instructions. *Ans.* £26·16.

Note.—To those students who are obtaining their initial experiences regarding computations of the order contained in this supplement, the following reminders may be helpful :

(1) If a given Principal is known to produce an Amount under certain conditions regarding " R " and "ⁿn," then, under similar conditions, other Principals and corresponding Amounts will vary in direct proportion.

(2) As regards compounding and amortization it is *wrong* to assume that proportional changes of " R " or " n " will have similar proportional effects on results.

Example 2.

In how many *years* **will £26 invested at $4\frac{3}{8}$% Compound Interest become £163 16s. 0d. ?**

$$£26 \text{ becomes } £163 \text{ 16s. 0d.}$$

$$£1 \text{ becomes } \frac{163\cdot 8}{26} \text{ Using C and D Scales } = 6\cdot 30$$

$$\text{Then } 6\cdot 3 = \left(1 + \frac{4\frac{3}{8}}{100}\right)^n$$

To determine "*n*" **(when the given R is less than 10%).**

1. Move the slide so that the index Y is at $4\frac{3}{8}$% on the Differential Scale.

2. Move the cursor to 4·375 on the C Scale.

3. According to whether the 6·3 on the Log-log Scale lies to the left or right of the cursor bring 10 or 1 of the C Scale respectively to the cursor.

In this case bring C 10 to the cursor.

4. Move the cursor to 6·3 on the Log-log Scale and read the significant figures of the number of years on the C Scale, viz., 4296, experience fixes the d.p. at 42·96 or, approximately, **43** years.

Example 2a.

In how many years will £26 invested at 17% Compound Interest become £163 16s. 0d. ?

£26 becomes £163·8 then, £1 becomes £6·3.

$$\left(1 + \frac{17}{100}\right)^n = 6·3.$$

To determine "n" (when R is greater than 10%). (In such cases the Differential Scale is not called upon to function.)

1. Set the cursor to 1·17 on the Log-log Scale.

2. Move the slide bringing 1 (or 10) of the C Scale to the cursor.

3. Move the cursor to 6·3 on the Log-log Scale, then the *number of years* is the C Scale reading at the cursor, *i.e.*, *11·72 years.*

Example 3.

What *Rate of Interest* over a period of 12 years would produce an amount of £67 4s. 0d. from a Principal of £35 ?

£35 becomes £67 4s. 0d.

£1 becomes $\dfrac{67·2}{35}$ (Use C and D Scales) = 1·92.

Then $1·92 = \left(1 + \dfrac{R}{100}\right)^{12}$

or $\left(1 + \dfrac{R}{100}\right) = \sqrt[12]{1·92}$

To determine R.

1. Set the cursor to 1·92 on the Log-log Scale.

2. Move the slide so as to bring 12 on the C Scale opposite the cursor.

3. Move the cursor to 1 (or 10) on the C Scale (in this case 1).

4. Move the slide so that the Index Z reads on the Inverse L.L. Differential Scale approximately the same as does the cursor on D scale.

The significant figures of the Rate of Interest is then the C Scale reading at the cursor viz. : 5·59%.

Note.—

(Operation 4 is applicable when the rate in question is normal and lies between 1% and 10%; for abnormal rates, *i.e.*, higher than 10%, take the Log-log Scale reading at the cursor after operation 3. Such an answer as 1·24 would then represent 24%. In this regard the user's experience and discretion (by rough approximation) must obtain.

Example 4. (Where R × n is less than 10)

Calculate the amount of £1 for a period of 3 years at 2¾% Compound Interest.

$$\text{Here A} = \left(1 + \frac{2\frac{3}{4}}{100}\right)^3 = \left(1 \cdot 0275\right)^3$$

To evaluate $(1 \cdot 0275)^3$

Obtain the value of ·0275 × 3 = ·0825

By rough approximation

$$(1 \cdot 0275)^3 \fallingdotseq 1 \cdot 0825$$

i.e., it is within the range 1·1 to 1·01 *i.e.*, the 1st remove.

For exact determination—

Bring Index Y to 2¾% on the Direct Log-log Differential Scale and K to C825 ; without disturbing the cursor take the D reading at K, *viz.:* 814 and move the slide so that the index Z is at 8·14 on the Inverse Log-log Scale.

Take the C scale reading at K, *viz.:* 848

Then $(1 \cdot 0275)^3 = 1 \cdot 0848$

Alternative method—

1. Move the slide so that the Index Y is at $2\frac{3}{4}\%$ on the Differential Scale.

2. Set the cursor at 2·75 on the C Scale.

3. Move 1 (or 10) on the C Scale to the cursor

4. Move the cursor to 3 (years) on the C scale.

5. Move the slide so that the index Z reads the same on the Inverse L.L. Differential Scale as does the D scale at the cursor, and take the C scale reading which will be found to be 8·48.

$$\text{Then Amount} = 1 + \frac{8·48}{100} = \text{£1·0848}$$

Example 4a. **(For abnormal rates, R greater than 10%).**
Calculate the amounts of £1 invested at 12% after periods of 3, 7, and 14 years respectively.

Note.—In such cases the Differential Scale is not called upon to function.

1. Set the cursor to $\left(1 + \dfrac{12}{100}\right)$ *i.e.* 1·12 on the upper Log-log Scale.

2. Move the 1 (or 10) of the C Scale to the cursor.

3. (*a*) Move the cursor to 3 on the C Scale and read 1·405 on the upper Log-log Scale.

    (*b*) Move the cursor to 7 on the C Scale and read 2·21 on the upper Log-log Scale.

    (*c*) Move the cursor to 14 on the C Scale and read 4·89 on the lower Log-log Scale.

    Ans. £1·405, £2·21 and £4·89.

5. Calculate the equal annual " Loan Charge " to repay a Loan of £100 in 15 years, the Interest being at the rate of $4\frac{1}{4}\%$.

$$\text{The Loan Charge per £1} = \frac{\dfrac{4\frac{1}{4}}{100}\left(1 + \dfrac{4\frac{1}{4}}{100}\right)^{15}}{\left(1 + \dfrac{4\frac{1}{4}}{100}\right)^{15} - 1} \quad \text{(Formula 5)}$$

$$\left(1 + \frac{4\frac{1}{4}}{100}\right)^{15} \text{ evaluated as in Example 1} = 1\cdot867.$$

Then charge per £1 $= \dfrac{\cdot0425 \times 1\cdot867}{1\cdot867 - 1}$

$$= \frac{\cdot0425 \times 1\cdot867}{\cdot867} \quad \begin{array}{l}\text{Use C and D Scales} \\ \text{to evaluate}\end{array}$$

$$= \cdot0915$$

Charge per £100 $= 9\cdot15$, *i.e.* £9 3s. 0d.

6. What equal annual amount set aside at the end of each of 15 years would accumulate at 3% to repay, at the close of the period, the Principal and Interest of a $4\frac{1}{4}$% Loan of £54.

Sinking Fund Accumulation per £1 per annum for 15 years at 3% $\left.\vphantom{\begin{array}{c}a\\b\\c\end{array}}\right\} = \dfrac{\left(1 + \dfrac{3}{100}\right)^{15} - 1}{\dfrac{3}{100}}$ (Formula 3)

$$\left(1 + \frac{3}{100}\right)^{15} \text{ evaluated as in Example 1} = 1\cdot558$$

S.F. Accumulation per £1 $= \dfrac{1\cdot558 - 1}{\cdot03} = \dfrac{\cdot558}{\cdot03}$

$$= 18\cdot6$$

Amount of £1 for 15 years at $4\frac{1}{4}$% $= \left(1 + \dfrac{4\frac{1}{4}}{100}\right)^{15}$ (Formula 1)

(Evaluated as in Example 1) $= 1\cdot867$

Then the Amount to be set aside each year $= \dfrac{54 \times 1\cdot867}{18\cdot6}$

(Use C and D Scales to evaluate) $=$ £5 38

Amortization of Assets
and Departmental Charges.

7. A machine, purchased at £195, after functioning for 9 years had a market value of £43.

For machines of this order, apart from maintenance and assuming current interest rate at $3\frac{3}{4}\%$—

(a) What depreciation percentage should apply for intermediate valuations of assets ?

(b) What departmental annual charge should be *debited* for the use of the machine ?

Part *(a)*—

£195 in 9 years depreciates to £43.

Reverse and consider a growth series, namely :

£43 in 9 years becomes £195 (See note *re* Formulæ 2 & 2a)

Then £1 in 9 years becomes $\dfrac{195}{43} = 4\cdot53$

Now estimate whether the growth rate applicable to this series is above or below 10%, by determining the value of £1 after 9 years at 10%. If this computation, effected as explained in Example 1, gives a figure in excess of 4·53 then the growth rate applicable is *less* than 10%; if below 4·53 the growth rate is *above* 10%. (Amortization rates applicable to manufacturing plant are usually in excess of 10%).

$$\text{Amount of £1 after 9 years at 10\%} = \left(1 + \frac{10}{100}\right)^9$$

(Evaluated as in Example 1 = 2·36)
i.e., Applicable growth rate is *above* 10%.

Proceed to determine the $\sqrt[9]{4\cdot53}$ as follows.—

1. Set the cursor at 4·53 on the Log-log Scale.

2. Bring 9 on the C Scale to the cursor.

3. Move the cursor to 10 (or 1) on the C Scale and read 1·183 on the upper Log-log Scale.

Then $\left(1 + \dfrac{R}{100}\right) = 1·183.$ R = 18·3%

Then Depreciation percentage $= \dfrac{18·3}{1·183}$ (Formula 2)

$= 15·47$ *Say* 15½%

Part (b)

Amount of £1 for 9 years at 3¾% $= \left(1 + \dfrac{3\frac{3}{4}}{100}\right)^{9}$

(Formula 1)

(Evaluated as in Example 1) $= 1·393$

Value of £195 after 9 years at 3¾% $= 195 \times 1·393$,

(Use C and D Scales)

$= 271·6$

Sinking Fund Accumulation of £1 per annum for 9 years at 3¾% $\left.\right\} = \dfrac{1·393-1}{·0375}$ (Formula 3)

$= 10·47$

Annual Departmental Charge $= \dfrac{271·6-43}{10·47}$

$= \dfrac{228·6}{10·47}$

(Use C and D Scales)

$= £21·83$

8. A motor car, purchased at £240, after 5 years has a market value of (say) £25. Assuming a current rate of interest at 4½% determine :

(a) The " annual premium on ownership" that this represents.

(b) The percentage rate of amortization.

£1 for 5 years at $4\frac{1}{2}\% = \left(1 + \dfrac{4\frac{1}{2}}{100}\right)^{5} = 1\cdot246$ $\begin{cases}\text{Formula 1}\\\text{Example 1}\end{cases}$

£240 for 5 years at $4\frac{1}{2}\% = 1\cdot246 \times 240 = £299$

(C and D Scales)

"Annual premium in ownership" $= \dfrac{299 - 25}{\dfrac{1\cdot246 - 1}{\cdot045}} = \dfrac{274}{5\cdot47}$

$= £50\cdot1$ (C and D Scales)

For part (b) reverse and consider growth from £25 to £240.

Then $\left(1 + \dfrac{R}{100}\right)^{5} = \dfrac{240}{25}$, $\sqrt[5]{9\cdot6} = 1\cdot572$ (See Ex. 3)

$\therefore R = 57\cdot2\%$

(b) The percentage rate of amortization $D = \dfrac{57\cdot2}{1\cdot572}$

$= 36\cdot4\%$

9. Fourth Issue Savings Certificates cost 16/- each and reach a value of 23/- on the expiration of 11 years. (Interest increments not subject to Income Tax.)

Determine the equivalent rate relative to normal taxed dividends that this represents to—

(a) The non-taxpayer. (b) Persons taxed at 2s. 6d.

(c) Persons taxed at 5/-.

$$\left(1 + \dfrac{R}{100}\right)^{11} = \dfrac{23}{16} = 1\cdot438$$

$$\text{Then } \left(1 + \dfrac{R}{100}\right) = \sqrt[11]{1\cdot438}$$

Determine R as explained in Ex. 3 $= 3\cdot357\%$

(a) Rate to non-taxpayer $= 3\cdot357\%$

(b) Equivalent to taxpayer at 2s. 6d. $= 3\cdot84\%$

(c) Equivalent to taxpayer at 5s. 0d. $= 4\cdot48\%$

Practice Exercises with hints.
Compound Growth.

10. Determine the AMOUNT of— *See*

(a) A Principal of £1 at the end of 15 years at $4\frac{3}{8}\%$ | Ex. 1a
(b) A Principal of £3 10s. at the end of 29 years at $3\frac{1}{4}\%$ | Ex. 1a
(c) A Principal of £151 at the end of 4 years at 14% | Ex. 4a

Ans. (a) £1·901. (b) £8·85. (c) £255.

11. Determine the PRINCIPAL that will produce—

(a) An Amount of £74 in 23 years at $3\frac{1}{2}\%$... ... | Ex. 1b
(b) An Amount of £519 in 45 years at $5\frac{1}{4}\%$ ... | Ex. 1b
(c) An Amount of £341 in 5 years at 12% ... ... | Ex. 4a

Ans. (a) £33·54. (b) £51·9. (c) £193·5.

12. In what TIME will—

(a) £32 become £50 at $4\frac{3}{4}\%$? ... ... | Ex. 2
(b) £49 5s. become £100 at $3\frac{3}{4}\%$? ... ... | Ex. 2
(c) £65 6s. become £640 at $22\frac{1}{2}\%$? ... ... | Ex. 2a

Ans. (a) 9·62 years. (b) 19·24 years. (c) 11·25 years.

13. What RATE PER CENT. will produce—

(a) £49 from £15 in 25 years? ... ... | Ex. 3
(b) £290 from £264 in 4 years?... ... ... | Ex. 3
(c) £380 from £120 in 8 years? ... ... | Ex. 3 }
Ans. (a) 4·85%. (b) 2·375% (c) 15·5% *Seenote* }

Amortization—Residue.

14. An amortization rate " D " of $8\frac{1}{2}\%$ is applied, at the end of each of 8 years, to an initial asset of £28. Determine the Residue at the end of the period.

1. Determine the corresponding R from—

$$R = \frac{D}{1 - \dfrac{D}{100}} = \frac{8 \cdot 5}{\cdot 915} \quad - \quad - \quad \text{(Use C and D Scales)}$$

2. Evaluate $\left(1 + \dfrac{R}{100}\right)^{8}$ - - - - - (As Example 1.)

3. Residue $= \dfrac{28}{\left(1 + \dfrac{R}{100}\right)^{8}}$ - - (Use C and D Scales.)

$Ans.$ £13·76

15. Determine the percentage of an orginal asset that remains after writing off 7% at the end of each of 12 years.

1. Obtain R from $R = \dfrac{D}{1 - \dfrac{D}{100}}$ - - - (Formula 2a)

2. Evaluate $\left(1 + \dfrac{R}{100}\right)^{12}$ - - - - - (Example 1)

Then Residue per £100 $= \dfrac{100}{\left(1 + \dfrac{R}{100}\right)^{12}}$

$Ans.$ 41·86%

16. What remains of a £200 asset after writing off 16% at the end of each of 5 years?

1. $R = \dfrac{16}{1 - ·16} = \dfrac{16}{·84}$ - - - - - - (Formula 2a)

Evaluate, as in Example 4a, $\left(1 + \dfrac{R}{100}\right)$

Residue $= \dfrac{200}{\left(1 + \dfrac{R}{100}\right)^{5}}$

$Ans.$ £83·6

Amortization—Time.

17. In what time will an Asset of £230 write down to £40 at 6% per annum?

$$R = \dfrac{6}{1 - ·06} = \dfrac{6}{·94}$$

Then $\left(1 + \dfrac{R}{100}\right)^{n} = \dfrac{230}{40}$

Evaluate "n" as explained in Example 2. $Ans.$ 28·27 yrs.

18. In what time will an asset, subjected to an amortization rate of 23%, write down to 16% of its original value ?

Then 100 in " n " years becomes 16.

(1) Determine $R = \dfrac{23}{1-\cdot 23} = \dfrac{23}{\cdot 77}$ - - - - (Formula 2a)

$$\left(1 + \frac{R}{100}\right)^n = \frac{100}{16}$$

(2) Evaluate " n " as explained in Example 2a.

Ans. 7·01 yrs.

Amortization—Rate.

19. Calculate the amortization rate applicable where a £25 asset depreciates to £15 in 8 years.

Consider £15 growing to £25 in 8 years

$$\left(1 + \frac{R}{100}\right)^8 = \frac{25}{15} = 1\cdot 667$$

(1) Evaluate R as explained in Example 3.

(2) Amortization Rate $D = \dfrac{R}{1 + \dfrac{R}{100}}$

Ans. 6·185%

20. Calculate the annual rate of depreciation where an original asset of £190 has a value of £20 after 6 years.

Consider £20 growing to £190 in 6 years.

$$\left(1 + \frac{R}{100}\right)^6 = \frac{190}{20}$$

Evaluate R as explained in Example 3, observing special note.

Annual Depreciation applicable $= D = \dfrac{R}{\left(1 + \dfrac{R}{100}\right.}$

Ans. 31·3%

21. Determine the rate of interest, applied yearly, that is equivalent to 4 per cent. per annum applied monthly, compound interest being reckoned.

Here £1 at the end of 1 year becomes $\left(1 + \dfrac{\frac{4}{12}}{100}\right)^{12}$

Evaluate as explained in Ex. 4.

$$= 1\cdot0407$$

i.e. Equiv. R $= 4\cdot07\%$

⬛This result, obtained from a 10 inch Slide Rule, is decidedly more accurate than is obtainable from the use of four-figure tables which give $4\cdot20$; the value obtained using seven-figure tables being $4\cdot074$.

22. Determine the respective rates of interest, applied yearly, that are equivalent to 5 per cent. per annum, applied (*a*) monthly, (*b*) quarterly, and (*c*) half yearly, compound interest assumed.

Evaluate the following as explained in Ex. 4.

$$(a) \left(1 + \dfrac{\frac{5}{12}}{100}\right)^{12}, \quad (b) \left(1 + \dfrac{\frac{5}{4}}{100}\right)^{4} \left(1 + \dfrac{\frac{5}{2}}{100}\right)^{2}$$

Hence respective rates :

(*a*) $5\cdot12\%$.   (*b*) $5\cdot09\%$.   (*c*) $5\cdot06\%$.

Assurance.

23. A person, age 37 years, who took out a ten years £1,000 with bonus assurance policy, paid commencement of year premiums of £109 and at the expiration of ten years received £1,177.

Assume income tax abatement at 2s. 3d. in the £1, on part of the premium (maximum 7 per 100 of the sum assured), and that the rate of interest available during the period was $4\frac{7}{8}\%$.

Determine (1) The equivalent amount that the policy holder would have received if the premiums, less income tax abatement, had been invested and the interest re-invested at the rate mentioned, the holders' dividends being (a) untaxed, (b) taxed at 2s. 3d. in the £1, or (c) taxed at 4s. 6d. in the £1.

(2) The percentage of the premium on the basis of (1a), (1b) or (1c) which, to the policy holder, can be considered as cover for premature death risk.

Equivalent untaxed investment rate
$$
\begin{cases}
(a) = 4\tfrac{7}{8}\% \times \dfrac{20}{20} &= 4\cdot875\% \\[2mm]
(b) = 4\tfrac{7}{8}\% \times \dfrac{17\cdot75}{20} &= 4\cdot327\% \\[2mm]
(c) = 4\tfrac{7}{8}\% \times \dfrac{15\cdot5}{20} &= 3\cdot778\%
\end{cases}
$$

Equivalent sum to be invested = £109 − (70 at 2s. 3d.)
$$= £101\cdot125$$

Accumulated value of £1 commencement of year premiums after n years

$$= \left\{ \frac{\left(1 + \dfrac{R}{100}\right)^{n+1} - 1}{\dfrac{R}{100}} \right\} - 1 \quad \text{----- Formula 4}$$

Evaluate $\left(1 + \dfrac{4\cdot875}{100}\right)^{11}$, $\left(1 + \dfrac{4\cdot327}{100}\right)^{11}$ and $\left(1 + \dfrac{3\cdot778}{100}\right)^{11}$

as explained in Example 1, and substitute these values together with the appropriate R's in Formula 4 and obtain

Respective accumulations of £101·125 for

$$(a) = 101\cdot125 \times 13\cdot11 = 1326$$
$$(b) = 101\;125 \times 12\cdot72 = 1286$$
$$(c) = 101\cdot125 \times 12\cdot33 = 1247$$

Respective percentages of premium to cover premature death risk, etc., of

untaxed holder $\qquad (a) = \left(\dfrac{1326 - 1177}{1326}\right) \times 100 = 11\cdot23\%$

holder taxed at 2s. 3d. in £1

$$(b) = \left(\frac{1286 - 1177}{1286}\right) \times 100 = 8 \cdot 48\%$$

holder taxed 4s. 6d. in £1

$$(c) = \left(\frac{1247 - 1177}{1247}\right) \times 100 = 5 \cdot 61\%$$

Assurance and Endowment.

24. What premiums invested at the commencement of each of 15 consecutive years will, at compound interest, accumulate a sufficient sum to provide *end* of year incomes of £156 for the subsequent 8 years? Assume the rate of interest available to be $4\frac{1}{4}\%$ added annually.

Note.—At the end of the 15th year accumulations must equal the present worth of £156 for 8 years.

Present worth of £1 per annum for 8 years at $4\frac{1}{4}\%$ $= \dfrac{\left(1 + \frac{4\frac{1}{4}}{100}\right)^8 - 1}{\frac{4\frac{1}{4}}{100}\left(1 + \frac{4\frac{1}{4}}{100}\right)^8}$ - - (Formula 5a)

Evaluate $\left(1 + \dfrac{4\frac{1}{4}}{100}\right)^8$ as in Example 1 $= 1 \cdot 395$

" Present worth " of £156 for 8 years at $4\frac{1}{4}\%$ $= 156 \times \dfrac{\cdot 395}{\cdot 0425 \times 1 \cdot 395} = £1040.$

Accumulations of £1 commencement of year premiums at the end of 15 years $= \left\{\dfrac{\left(1 + \frac{R}{100}\right)^{n+1} - 1}{\frac{R}{100}}\right\} - 1$ - (Formula 4)

$$= \left\{\dfrac{\left(1 + \frac{4\frac{1}{4}}{100}\right)^{16} - 1}{\frac{4\frac{1}{4}}{100}}\right\} - 1$$

Evaluate $\left(1 + \dfrac{4\frac{1}{4}}{100}\right)^{16}$ as Example 1 = 1·946.

Accumulations of £1 etc. $= \dfrac{·946}{·0425} - 1 = 21·27$.

Allocation to accumulate to £1040 $= \dfrac{1040}{21·27} = £48·9$.

Deferred Payments.

25. Determine the equal end of quarter instalments that should be charged for a machine valued at £180, on the basis that all outstanding amounts are charged each quarter an interest of 2%; the total payments to be completed at the end of the 5th year.

Here the number of instalments and periods = 20.

$$\text{Instalment to repay £1 of original value} = \dfrac{\dfrac{R}{100}\left(1 + \dfrac{R}{100}\right)^n}{\left(1 + \dfrac{R}{100}\right)^n - 1} \qquad \text{Formula 5}$$

Evaluate $\left(1 + \dfrac{2}{100}\right)^{20}$ as in Example 1 = 1·486

$$\text{Quarterly Instalment for £180 original value} = \left(\dfrac{·02 \times 1·486}{1·486 - 1}\right) \times 180$$

$$= \dfrac{·02 \times 1·486 \times 180}{·486}$$

$$= £11·01$$

Building Society—Periods.

26. A house valued at £495 is bought through a Building Society on the following terms—

Deposit £48 12s. and subsequent payments of £3 2s. every 28 days. Interest at the rate of $5\frac{3}{4}$% charged on the balance owing at the beginning of each year and credit given for the total amount of subscriptions paid each year.

After how many payments will the Loan be cleared on the assumption that all charges are met to date ?

Note.—The terms do not grant any interest on the separate deposits pending their credit at the end of the year, the normal value of such interest at 3% is approximately :

$$\frac{3\cdot1 \times 3}{100} \times \frac{78}{13} \times \frac{20}{1} = \text{10s. 11d. per annum.}$$

The " period of repayment " for the Loan £L by a premium £P is the period where £L at compound interest equals the Sinking Fund accumulations due to £P per annum at the same rate of interest.

i.e.
$$L\left(1 + \frac{R}{100}\right)^n = P\left\{\frac{\left(1 + \frac{R}{100}\right)^n - 1}{\frac{R}{100}}\right\}$$

which gives
$$\left(1 + \frac{R}{100}\right)^n = \frac{P}{P - \frac{L \times R}{100}}$$

Assuming a year of 364 days, number of payments per annum $= \frac{364}{28} = 13$, P $= 13 \times 3\cdot1$, Loan L $= 495 - 48\cdot6$
$= 446\cdot4$, n $=$ number of years of 13 payments each.

$$\left(1 + \frac{5\frac{3}{4}}{100}\right)^n = \frac{13 \times 3\cdot1}{13 \times 3\cdot1 - \frac{446\cdot4 \times 5\frac{3}{4}}{100}} = \frac{40\cdot3}{14\cdot63} = 2\cdot753$$

Evaluate n as explained in Example 2.

n $= 18\cdot11$ years or $18\cdot11 \times 13 = 235\frac{1}{2}$ payments.

Short Lease Property.

27. A particular property, the lease of which has 18 years to run, yields a nett annual income of £120.

What is the value of this property on the basis of 6% interest per annum on the capital sum and $2\frac{1}{8}$% compound interest available on the annual sinking fund allocations set aside to furnish the capital sum at the termination of the lease ?

$$\text{Value of £1 income or year's purchase} = \frac{100\left[\left(1 + \frac{r}{100}\right)^n - 1\right]}{R\left[\left(1 + \frac{r}{100}\right)^n - 1\right] + r} \quad \text{(Formula 6)}$$

(1) Evaluate $\left(1 + \frac{2\frac{1}{8}}{100}\right)^{18}$ as in example 1

$$= 1 \cdot 460$$

$$\text{Years' Purchase} = \frac{100 \times \cdot 460}{6 \times \cdot 460 + 2\frac{1}{8}} = \frac{6}{2 \, 760 + 2 \cdot 125}$$

$$= \frac{46}{4 \cdot 885} = 9 \cdot 42$$

Value of Property = Year's Purchase × Income . (Formula 7)
$$= 9 \cdot 42 \times 120$$
$$= £1130 \cdot 4$$

Annual Allocation to Sinking Fund
$$= 120 - \frac{6}{100} \times \frac{1130 \cdot 4}{1}$$
$$= 120 - 67 \cdot 8 = \underline{£52 \cdot 2}$$

Use of Log-log scales and gauge mark L (arranged on the C scale at 2·3026) to determine the values of logarithms to the base "e" or logarithms to the base 10 for the ranges 1·1052 to 2·7183 to 10 to 22000.

The upper and lower log-log scales are so arranged on the rule as to enable the significant figures of logarithms to the base "e" to be read, with the aid of the cursor, *direct from* the D scale.

From the portion of the top scale 1·1052 to 2·7183, the decimal point for logarithms to the base "e" is immediately *before* the first significant figure of the D scale reading. From the portion of the lower log-log scale 2·7183 to 22000 the decimal point for logarithms to the base "e" is immediately *after* the first significant figure of the D scale reading.

To determine $\log_e 2$ and $\log_{10} 2$

Set the cursor at 2 on the top log-log scale and read 6930 at the cursor on the D scale.

Then $\log_e 2 = ·6930$.

Remembering that $\dfrac{\log_e N}{2·3026} = \log_{10} N$, bring gauge mark L (on the C scale at 2·3026) to the cursor and read 301 at C_1 or C_{10} (C_1 in this case) on the D scale.

Then $\log_{10} 2 = ·301$.

To determine $\log_e 4·55$ and $\log_{10} 4·55$

Set the cursor at 4·55 on the lower log-log scale and read at the cursor on the D scale 1515

Then $\log_e 4·55 = 1·515$

Move the slide bringing gauge mark L on the C scale to the cursor and read at C_1 or C_{10} (C_1 in this case) on the D scale 658.

Then $\log_{10} 4·55 = ·658$

To determine $\log_e 875$ and $\log_{10} 875$

Read from lower log-log scale and D scale

$$\log_e 875 = 6.774.$$

To determine $\log_{10} 875 \cdot$ use the form 8.75×10^2. Set the cursor to 8.75 on the lower log-log scale, bring L on the C scale to the cursor and read at C_1 or C_{10} (C_{10} in this case) on the D scale 942 the significant figures of the mantissa of the $\log_{10} 8.75$, which in conjunction with the characteristic 2 *i.e.* factor 10^2 gives

$$\log_{10} 875 - 2.942.$$

Special use of the **Direct Differential Scale** in conjunction with the C scale and gauge mark L to determine the values of logarithms to the base "e" or base 10 for the ranges

(1) 1.1052 to 1.01

(2) 1.01 to 1.001

(3) 1.001 to 1.0000

(1) Range 1.1052 to 1.01.

To determine $\log_e 1.035$ *i e.*, $\log_e \left(1 + \dfrac{3\frac{1}{2}}{100} \right)$, and $\log_{10} 1.035$

Move the slide so that the index Y is at $3\frac{1}{2}$% (or 1.035 upper numbering) on the Direct Differential Scale and set the cursor at 35 on the C scale. Read on the D scale, at the cursor, the significant figures of $\log_e 1.035$, namely 344. For this range the significant figures are preceded by *one cipher* before the decimal point.

$$i.e., \log_e 1.035 = .0344.$$

For $\log_{10} 1.035$ bring gauge mark L to the cursor line and read at C_1 or C_{10} (C_1 in this case) on the D scale 1494

$$\log_{10} 1.035 - \frac{.0344}{2.3026} = .01494.$$

(2) Range 1.01 to 1.001.

To determine $\log_e 1.005$ *i.e.* $\log_e \left(1 + \dfrac{\frac{1}{2}\%}{100} \right)$ and $\log_{10} 1.005$

Move the slide so that the index Y is at $\frac{1}{2}$% on the Direct

Differential Scale and the cursor at 5 on the C Scale. Read on the D Scale at the cursor the significant figures of $\log_e 1 \cdot 005$, namely 499.

For this range the significant figures are preceded by two ciphers before the decimal point.

$$\log_e 1 \cdot 005 = \cdot 00499$$

For $\log_{10} 1 \cdot 005$, bring gauge mark L to the cursor line and read at C_1 or C_{10} (C_1 in this case) on the D scale 217.

$$\log_{10} 1 \cdot 005 = \frac{\cdot 00499}{2 \cdot 3026} = \cdot 00217$$

(3) Range 1·001 to 1·000

Take $\log_e 1 \cdot 000N \backsimeq \cdot 000N$, and $\log_{10} 1 \cdot 000N = \dfrac{\cdot 000N}{2 \cdot 3026}$

$\log_e 1 \cdot 0000N \backsimeq \cdot 0000N$, and $\log_{10} 1 \cdot 0000N = \dfrac{\cdot 0000N}{2 \cdot 3026}$

and so on.

From these determinations of logarithms to the base "e" and base 10 for the three ranges, it will be seen that the "Differential Scale" links up the range of direct projection from the existing Log-log Scale with the region where logarithms to the base "e" are direct settings on the D Scale, so completing an equivalent $\log_e$ scale of infinite length.

Logarithms to the base 10 for the range 1·001 to 1·10, obtained by the application of the factor $\dfrac{1}{2 \cdot 3026}$ to the $\log_e$ values obtained from the Rule as previously described, will be found to be more accurate (when reference is made to seven-figure tables) than those obtainable from four-figure tables.

The following trials and comparisons will serve to illustrate this relative advantage of the " **Rule** " as compared with Four-figure logarithm tables.

Values obtained from	Log 1·01	Log 1·015	Log 1·03	Lg. 1·0475
Four-figure tables	·0043	·0064	·0128	·0202
The P.I.C. Slide-Rule	·00432	·00647	·01284	·02015
Seven-figure tables	·0043214	·0064660	·0128372	·0201540

As regards relative accuracy for finance computations, these trials furnish evidence of the advantages that the L.L. Differential Rule affords, inasmuch as logarithms included in the above range constitute the basis of commercial compounding calculations where *normal* interest rates apply.

Inverse L.L. Differential Scale.

This scale affords the same desirable advantage of positive setting in regard to reading of log-log extension values corresponding to given cursor positions as does the Direct L.L. Differential Scale in regard to placing the cursor to given log-log extension values. Its purpose will be readily understood from the following :

Regarding the powers of 10 and e

Take the case

$$10^{.7843} = e^{2.3026 \times .7843} = e^{1.806}$$

To obtain the value of $2.3026 \times .7843$, place C_1 or C_{10} at 7843 on D, move the cursor to L on the C scale and take the D scale reading 1806.

Assume the cursor at (say) 1806 on the D scale.

Then (i)$e^{1.806}$ or $10^{.7843}$

is the lower log-log scale reading at the cursor, $viz.$, 6 08.

(ii) The value of $e^{.1806}$ or $10^{.07843}$
is the upper log-log scale reading at the cursor, $viz.$, 1·198.

(iii The value of $e^{.01806}$ or $10^{.007843}$
$i.e.$. the log-log reading on the "1st remove" scale, move the slide so that the index Z is at $d = 1·8$ on the Inverse Differential Scale and note C scale reading at the cursor, $viz.$, 1822.
Then $e^{.01806}$ or $10^{.007843} = 1·01822$.
(Note the prefix 1·0 for the "1st remove")

(iv) The value of $e^{.001806}$ or $10^{.0007843}$
$i.e.$, the projected log-log reading on the "2nd remove" scale may be obtained as follows :
With the cursor still at 1806 on D, move the slide

so that the index Z is at $d = \cdot 1806$ on the Inverse Differential Scale and note C scale reading at the cursor *viz.*, 1808.

Then $e^{\cdot 001806} = 10^{\cdot 0007843} = 1\ 001808$

(Note the prefix 1·00 for the " 2nd remove")

As regards the degree of accuracy, the results obtained by means of the slide rule for such evaluations as (iii) and (iv) usually exceed that obtainable by the use of five figure logarithm tables.

Further examples of the use of the Inverse Differential Scale will be found in the instructions for examples **3** and **4** on pages 8 and 9 respectively.

Differential Scales.
Theoretical Considerations.

Differential Scales are selected Logarithmic Scales of $\dfrac{x}{f(x)}$ arranged for manipulation as divisor correctives in conjunction with a Logarithmic Scale of x in such a manner as to cancel out the x and leave the $f(x)$ operative :

$$i.e., \frac{x}{\dfrac{x}{f(x)}} = f(x)$$

The Differential Scale mode of applying a "Function" to the Slide Rule is advantageous where for a range of x under consideration the logarithmic scale length that is required for $\dfrac{x}{f(x)}$ is relatively short in comparison to that which would be required for $f(x)$.

The Direct and Inverse L.L. Differential Scales previously referred to which serve to furnish values of $\log_e (1 + x)$ and $e^x - 1$ are respectively logarithmic scales of $\dfrac{x}{\log_e(1 + x)}$ and $\dfrac{x}{e^x - 1}$